$$\int \cos \theta^2 \, d\theta = \theta \sum_{t=0} 2^t \prod_{s=1}^{s=t} \frac{1}{2s+1} \theta^{2t} \cos\left(t\frac{\pi}{2} - \theta^2\right)$$

$$= \cos\theta^2\left(\theta - \frac{2^2}{1\cdot3\cdot5}\theta^5 + \frac{2^4}{1\cdot3\cdot5\cdot7\cdot9}\theta^9 - + \cdots\right)$$

$$+ \sin\theta^2\left(\frac{2}{1\cdot3}\theta^3 - \frac{2^3}{1\cdot3\cdot5\cdot7}\theta^7 + - \cdots\right).$$

Two interesting special cases deserve mention.

When the exponent of the coefficient is less by unity than that of the argument of the function, the integral assumes a known form; and the value of the other integral becomes known. Hence, expressions of the type $\theta^{rn-1} \sin \theta^n \, d\theta$ are directly integrable in finite form by these formulæ. If in addition n takes the special form $1/r$, the integral of $\sin \theta^{1/r} d\theta$ is found, that is, the integral of sine or cosine of any root of a variable is found in finite form.

Thus, taking the first of the four formulæ, making $m = n - 1$, and transforming a little; and after this, making $n = 1/r$, we obtain the formulæ:

$$\int \theta^{rn-1} \cos\left(r\frac{\pi}{2} - \theta^n\right) d\theta = - n^{-1} \prod_{s=r-1}^{s=1} s \cdot \sum_{t=1}^{t=r} \prod_{s=1}^{s=t-1} s^{-1} \cdot \theta^{(t-1)n} \sin\left(t\frac{\pi}{2} - \theta^n\right),$$

$$\int \cos\left(r\frac{\pi}{2} - \theta^{1/r}\right) d\theta = - r \prod_{s=1}^{s=r-1} s \cdot \sum_{t=1}^{t=r} \prod_{s=1}^{s=t-1} s^{-1} \cdot \theta^{(t-1)1/r} \sin\left(t\frac{\pi}{2} - \theta^{1/r}\right).$$

As examples, in these formulas making $n = 2$, $r = 4$, we have

$$\therefore \quad \int \theta^7 \cos\theta^2 \, d\theta = -3\left[\cos\theta^2\left(1 - \frac{\theta^4}{2}\right) + \sin\theta^2\left(\theta^2 - \frac{\theta^6}{3!}\right)\right],$$

$$\int \cos\theta^{1/4} = -4!\left[\cos\theta^{1/4}\left(1 - \frac{\theta^{1/2}}{2}\right) + \sin\theta^{1/4}\left(\theta^{1/4} - \frac{\theta^{3/4}}{3!}\right)\right].$$

DISCUSSIONS.

I. Relating to a Curve with Unusual Properties.

By Jos. B. Reynolds, Lehigh University.

The curve

$$y = \frac{a^2 x}{x^2 + a^2}$$

presents some unusual properties. Considering the part of the curve to the right of the y-axis (see the figure) we have for the area between it and the x-axis

$$A = \int_0^\infty y\,dx = a^2 \int_0^\infty \frac{x\,dx}{x^2 + a^2} = \frac{a^2}{2} \log(x^2 + a^2)\Big]_0^\infty = \infty.$$

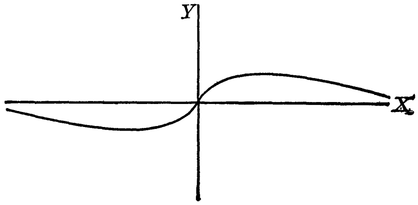
Yet the volume generated by revolving this area about the x-axis is

$$V = \pi \int_0^\infty y^2 dx = \pi a^4 \int_0^\infty \frac{x^2 dx}{(x^2 + a^2)^2} = \frac{\pi^2 a^3}{4};$$

that is, the revolution of an infinite area gives a finite volume. Again, the ordinate of the center of gravity of this area is

$$\bar{y} = \frac{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} ydy}{\int_0^\infty dx \int_0^{a^2x/(x^2+a^2)} dy} = \frac{\frac{\pi a^3}{8}}{\infty} = 0;$$

that is, although the area lies entirely above the x-axis and is indefinitely large,



its center of gravity is indefinitely close to that axis.

By the theorem of Pappus we have

$$2\pi\bar{y}A = V,$$

whence we have an indeterminate form evaluated

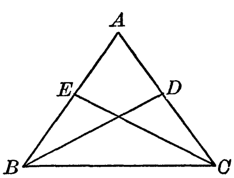
$$2\pi(0)\infty = \frac{\pi^2a^3}{4}.$$

II. Relating to a Demonstration of a Geometrical Theorem.

By William E. Heal, Washington, D. C.

If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

This is a very celebrated theorem and has been demonstrated in many ways. It was proposed in the Monthly as problem 42 and several demonstrations were published in Vol. II, pp. 157 and 189–191. Of these proofs all but one were by indirect methods; that is, by use of the *reductio ad absurdum.* The following direct method of proof was communicated to the writer some months ago by Dr. Artemas Martin of the U. S. Coast Survey and seems to be eminently worthy of preservation.



If $BD = CE$ we are to prove that $AB = AC$.

We have by a well-known theorem

$$AB \times BC = (AE + BE)BC = BD^2 + AD \times DC, \quad (1)$$

$$AC \times BC = (AD + DC)BC = CE^2 + AE \times EB. \quad (2)$$

If $BD = CE$ these become

$$(AE + BE)BC = BD^2 + AD \times DC, \quad (3)$$

$$(AD + DC)BC = BD^2 + AE \times EB. \quad (4)$$